MATH2230 HW1 Solution

P5

(5) let $z_1 = a+bi$, $z_2 = c+di$, $a,b,c,d \in \mathbb{R}$

$$z_1 z_2 = (a+bi)(c+di)$$
$$= ac + adi + bci - bd$$
$$= (ac-bd) + i(ad+bc)$$

$$z_2 z_1 = (c+di)(a+bi)$$
$$= ca + cbi + dai - db$$
$$= (ac-bd) + i(ad+bc)$$ (by commutative of real numbers).

$\therefore z_1 z_2 = z_2 z_1$

(6) a let $z_1 = a+bi$, $z_2 = c+di$, $z_3 = e+fi$, $a,b,c,d,e,f \in \mathbb{R}$

$$(z_1+z_2)+z_3 = (a+c)+i(b+d) + e+fi$$
$$= (a+c+e) + i(b+d+f)$$
$$= z_1 + (z_2+z_3)$$ (by associative of real numbers)

b $$z_1(z_2+z_3) = (a+bi)((c+e)+(d+f)i)$$
$$= a(c+e) + a(d+f)i + b(c+e)i - b(d+f)$$
$$= ac+ae-(bd+bf) + i(ad+af+bc+be)$$
$$z_1z_2 + z_1z_3 = (a+bi)(c+di) + (a+bi)(e+fi)$$
$$= z_1(z_2+z_3).$$

P8

(6) $$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \left[z_1\left(\frac{1}{z_3}\right)\right]\left[z_2\left(\frac{1}{z_4}\right)\right]$$ by ((10): $\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right)$)

$$= (z_1 z_2)\left(\frac{1}{z_3}\right)\left(\frac{1}{z_4}\right)$$ by associative law

$$= \frac{z_1 z_2}{z_3 z_4}$$ by ((10) and (11): $\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = \frac{1}{z_1 z_2}$)

(8) It is clearly true if $n=1$.

Assume that it is true for $n=m$

when $n=m+1$,

$$(z_1+z_2)^{m+1} = (z_1+z_2)(z_1+z_2)^m$$

$$= (z_1+z_2)\sum_{k=0}^{m} C_k^m z_1^k z_2^{m-k}$$

$$= \sum_{k=0}^{m} C_k^m z_1^{k+1} z_2^{m-k} + \sum_{k=0}^{m} C_k^m z_1^k z_2^{m-k+1}$$

$$= z_2^{m+1} + \sum_{k=0}^{m} \left(C_k^m + C_{k-1}^m\right) z_1^k z_2^{m+1-k} + z_1^{m+1}$$

Since $C_k^m + C_{k-1}^m = \frac{m!}{(m-k)!k!} + \frac{m!}{(m-(k-1))!(k-1)!} = \frac{(m+1)!}{(m-k+1)!\,k!} = C_k^{m+1}$

the result follows.

P13-P14

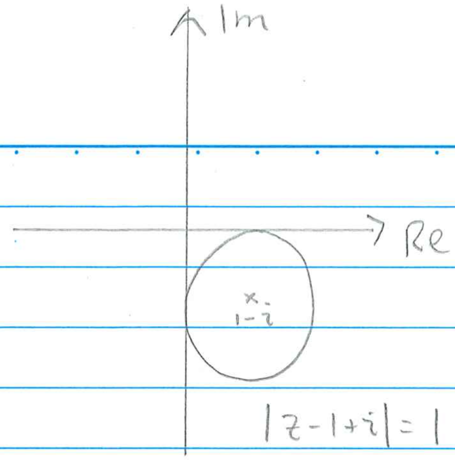
(4) let $z = x+yi$

$$(\sqrt{2}|z|)^2 = 2|z|^2 = 2x^2+2y^2$$

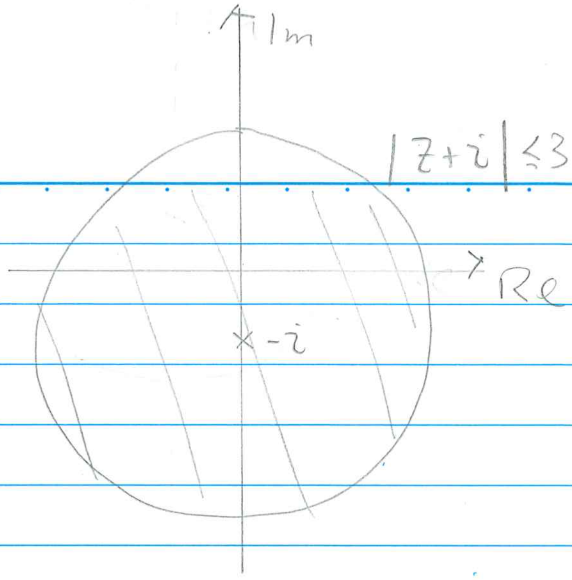
$$(\sqrt{2}|z|)^2 - (|\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2 = x^2+y^2-2xy$$

$$= (x-y)^2 \geq 0.$$

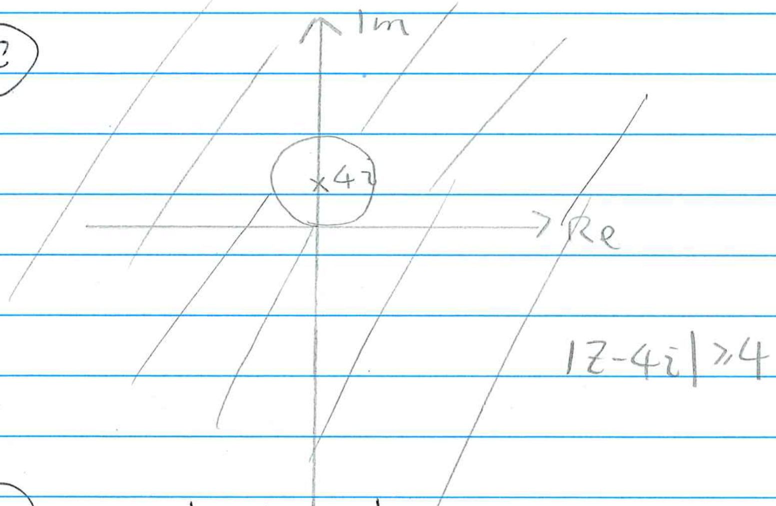
5 (a)



(b)



(c)



6 $|z-1| = |z+i|$ represents the perpendicular bisector of line segment joining 1 and $-i$.

8 $$|(x_1+iy_1)(x_2+iy_2)| = |(x_1x_2-y_1y_2)+i(x_2y_1+x_1y_2)|$$
$$= \left[(x_1x_2-y_1y_2)^2+(x_2y_1+x_1y_2)^2\right]^{1/2}$$
$$= \left(x_1^2x_2^2+y_1^2y_2^2+x_2^2y_1^2+x_1^2y_2^2\right)^{1/2}$$
$$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

P 23-24

1 a $z = \dfrac{-2}{1+\sqrt{3}i} = \dfrac{-2(1-\sqrt{3}i)}{2} = -1+\sqrt{3}i$

$\mathrm{Arg}(z) = \tan^{-1}(-\sqrt{3}) = 2\pi/3$

b $\mathrm{Arg}(\sqrt{3}-i) = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = -\pi/6 \Rightarrow \mathrm{Arg}\left[(\sqrt{3}-i)^6\right] = -\pi \;(\pi)$

5 a $i = e^{\pi/2 i}$, $(1-\sqrt{3}i) = 2e^{-\pi/3 i}$, $\sqrt{3}+i = 2e^{\pi/6 i}$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{\pi/3 i} = 2(1+\sqrt{3}i)$$

b $i = e^{\pi/2 i}$ $\quad 2+i = \sqrt{5}e^{\theta i}$ $\quad \theta = \tan^{-1}\frac{1}{2}$

$$\frac{5i}{2+i} = \sqrt{5}e^{(\pi/2-\theta)i}$$

$$= \sqrt{5}e^{(\tan^{-1}2)i}$$

$$= 1+2i$$

c $\sqrt{3}+i = 2e^{\pi/6 i} \Rightarrow (\sqrt{3}+i)^6 = -64$

d $(1+\sqrt{3}i) = 2e^{\pi/3 i} \Rightarrow (1+\sqrt{3}i)^{-10} = 2^{-10}e^{-10\pi/3 i}$

$$= 2^{-10}e^{2\pi/3 i}$$

$$= 2^{-11}(-1+\sqrt{3}i)$$

9 $(1+z+z^2+\dots+z^n)(1-z) = 1+z+\dots+z^n-(z+z^2+\dots+z^{n+1})$

$$= 1-z^{n+1}$$

The identity follows.

let $z = e^{i\theta} = \cos\theta + i\sin\theta$

L.H.S. $= 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta}$

Real part of L.H.S. $= 1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta$.

R.H.S. $= \dfrac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \dfrac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{(1-\cos\theta - i\sin\theta)} \cdot \dfrac{1-\cos\theta + i\sin\theta}{1-\cos\theta + i\sin\theta}$

Real part of R.H.S. $= \dfrac{(1-\cos(n+1)\theta)(1-\cos\theta) + \sin(n+1)\theta\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta}$

$$= \frac{1-\cos\theta-\cos(n+1)\theta+\cos\theta\cos(n+1)\theta+\sin(n+1)\theta\sin\theta}{2-2\cos\theta}$$

$$= \frac{1}{2}+\frac{-\cos(n+1)\theta+\cos n\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2}+\frac{\sin(2n+1)\theta/2}{2\sin\theta/2}$$

(10) (a) $(e^{i\theta})^3 = (\cos\theta+i\sin\theta)^3$

$$e^{3i\theta} = \cos^3\theta+3i\sin\theta\cos^2\theta-3\sin^2\theta\cos\theta-i\sin^3\theta$$

$\cos 3\theta + i\sin 3\theta = \cdots$

Done by comparing real and imaginary part